On a generalization of Valiron's inequality for k-hypermonogenic functions on upper half-space

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Abstract. We present some results on the asymptotic growth behavior of periodic k-hypermonogenic functions on upper half-space. A generalization of the classical Valiron inequality for this class of functions and some basic properties are discussed.

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INTRODUCTION

For details on Clifford algebras and its function theory we refer for example to [2, 9]. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and Cl_n be the associated real Clifford algebra obtained from the generating relations $e_ie_j + e_je_i = -2\delta_{ij}$, $i, j = 1, \cdots, n$ where δ_{ij} is the Kronecker symbol. Each $a \in Cl_n$ has the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$, $A \subseteq \{1, \cdots, n\}$, $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \le l_1 < \cdots < l_r \le n$, and $e_\emptyset =: e_0 =: 1$. The Clifford main involution is defined by $e_0' = 1$ and $e_i' = -e_i$ for $i = 1, \cdots, n$. The involution is defined by $\widehat{e_n} = -e_n$ and $\widehat{e_i} = e_i$ for $i = 0, \cdots, n-1$. For any arbitrary $a, b \in Cl_n$ and $c \in Cl_{n-1}$ one has $(ab)' = a'b'$, $\widehat{ab} = \widehat{a}\widehat{b}$, $a'e_n = e_n\widehat{a}$, $e_na' = e_n\widehat{a}$, $c'e_n = e_nc$, $e_nc' = ce_n$. The Clifford conjugate of a is defined by $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e_A} = \overline{e}_{l_r}\overline{e}_{l_{r-1}} \cdots \overline{e}_{l_1}$ and $\overline{e}_j = -\overline{e}_j$, $j = 1, \ldots, n$, $\overline{e}_0 = e_0 = 1$. Any element $a \in Cl_n$ may be uniquely decomposed as $a = c + de_n$ for $c, d \in Cl_{n-1}$. On the basis of this definition one defines the mapping $P : Cl_n \to Cl_{n-1}$ and $Q : Cl_n \to Cl_{n-1}$ by $P(a) = b$ and $Q(a) = d$.

The linear subspace $\mathrm{span}_{\mathbb{R}}\{1, e_1, \cdots, e_n\} = \mathbb{R} \oplus \mathbb{R}^n \subset Cl_n$ is the so-called space of paravectors $v = x_0 + x_1e_1 + x_2e_2 + \cdots + x_ne_n$ which we simply identify with $\mathbb{R}^{n+1}$. The element $x_0 =: Sc(v)$ is called the scalar part of the paravector v while the expression $z := x_1e_1 + \cdots + x_ne_n =: Vec(v)$ is called its vector part. We further indicate its $*$-invariant part by $\underline{z} := x_1e_1 + \cdots + x_{n-1}e_{n-1}$.

The aim of this paper is to study some questions related to the growth behavior of Clifford valued functions $f(z) = \sum_A e_A f_A(z)$, that satisfy the equation

$$(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} e_i) f(z) + \frac{k}{x_n} Q' f(z) = \beta f(z), \quad k \in \mathbb{R},\ \beta \in \mathbb{R}_0^+ \tag{1}$$

on upper half-space $H^+(\mathbb{R}^n) = \{z = e_1x_1 + \cdots + e_nx_n \in \mathbb{R}^n \mid x_n > 0\}$ for general $k \in \mathbb{R}$ and general $\beta \in \mathbb{R}_0^+$.

Notice if $\beta = 0$ the solutions to this equation are called k-hypermonogenic functions and are studied by a constantly growing community, see for instance [1, 11, 12, 14]. Furthermore, for $\beta = 0$ and $k = 0$ the expression (1) is the generalized Cauchy-Riemann equation where the solutions are called monogenic functions and in the case $\beta = 0$ where $k = n-1$ the expression (1) is the Dirac-Hodge equation and the solutions are called hypermonogenic functions. The Dirac-Hodge operator is a linearization of the Laplace-Beltrami operator on the upper half-space. This paper provides an extension to [6] where the particular case $k = n-1$, $\beta = 0$ has been treated.

In what follows, we use $\mathbf{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{N}_0^{n-1}$ to be an $(n-1)$-dimensional multi-index and $|\mathbf{m}| := m_1 + \cdots + m_{n-1}$.

FOURIER INTEGRAL REPRESENTATION

The study of the Taylor coefficients of an entire monogenic solution was the basis for the asymptotic growth analysis of entire solutions to partial differential equations (PDEs) related to the generalized Euclidean Dirac and Cauchy-Riemann operator, see for example [3, 4, 5, 8]. In the context of hypermonogenic functions defined on the upper half-space the study of the Taylor coefficients is replaced by studying their Fourier images, see [6].

Note that one can represent any upper half-space solution to (1) that additionally satisfies

$$\int_{\mathbb{R}^{n-1}} \|f(z)\| dx_1 \cdots dx_{n-1} < \infty \tag{2}$$

by the following Fourier integral

$$f(z) = \int\limits_{\mathbb{R}^{n-1}} \alpha(\underline{w}, x_n) e^{i\langle \underline{w}, \underline{x} \rangle} dw_1 \cdots dw_{n-1}, \tag{3}$$

where $\underline{x} = x_1 e_1 + \cdots + x_{n-1} e_{n-1}$ and $\underline{w} = w_1 e_1 + \cdots + w_{n-1} e_{n-1}$.

In what follows K_s, for $s \in \mathbb{R}$, denote the usual modified Bessel functions. For its definition and properties, see for instance [13].

Theorem 1 *Let $f : H^+(\mathbb{R}^n) \to Cl_n$ be a solution to (1) satisfying (2). Then for $\underline{w} \neq 0$ the Fourier images of the Fourier integral representation (3) have the following form*

$$\alpha(\underline{w}, x_n) \quad = \quad \phi(\underline{w}) \tilde{\alpha}(\beta + \|\underline{w}\|, x_n) \tag{4}$$

where precisely

$$\tilde{\alpha}(y, x_n) = (y x_n)^{\frac{k+1}{2}} \left(K_{\frac{k+1}{2}}(y x_n) + e_n K_{\frac{k-1}{2}}(y x_n) \right) \tag{5}$$

and $\phi(\underline{w})$ is a Cl_{n-1} valued function.

Proof. Using the same arguments given in [7], we consider the equation

$$(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} e_i) f(z) + \frac{k}{x_n} Q' f(z) = \beta f(z), \quad \beta \in \mathbb{R}_0^+.$$

Applying the Fourier transform in the first $(n-1)$ components leads to the following differential equation on the Fourier image $\alpha = \alpha(\underline{w}, x_n)$, $\underline{w} \in \mathbb{R}^{n-1}$, $x_n > 0$:

$$(i\underline{w} - \beta)\alpha + e_n \frac{\partial \alpha}{\partial x_n} + \frac{k}{x_n} Q' \alpha = 0.$$

Splitting α into its P-part and Q-part $\alpha = P\alpha + e_n Q' \alpha$ one has

$$(i\underline{w} - \beta) \left(P\alpha + e_n Q'\alpha \right) + e_n \frac{\partial}{\partial x_n} \left(P\alpha + e_n Q'\alpha \right) + \frac{k}{x_n} Q'(P\alpha + e_n Q'\alpha) = 0. \tag{6}$$

Since α is a solution to (1), collecting in (6) all terms that include or do not include the factor e_n, one obtains

$$(i\underline{w} - \beta) P\alpha - \tfrac{\partial}{\partial x_n} Q'\alpha + \tfrac{k}{x_n} Q'\alpha = 0, \qquad e_n(-i\underline{w} - \beta) Q'\alpha + e_n \tfrac{\partial}{\partial x_n} P\alpha = 0 \ . \tag{7}$$

Decomposing $\alpha = A\alpha + \underline{w} B\alpha$ where A and B are appropriately Clifford operators, projecting α to the part of the Clifford algebra which intersects the subspace generated by $\underline{w}$ at 0, follows

$$P\alpha = A(P\alpha) + \underline{w} B(P\alpha), \qquad Q'\alpha = A(Q'\alpha) + \underline{w} B(Q'\alpha).$$

Using this decomposition in $P\alpha$ and $Q'\alpha$ one rewrites the system (7) in the following way

$$\frac{\partial}{\partial x_n} Q'\alpha \quad = \quad \frac{\partial}{\partial x_n} A(Q'\alpha) + \underline{w} \frac{\partial}{\partial x_n} B(Q'\alpha) = (i\underline{w} - \beta) A(P\alpha) - (i\|\underline{w}\|^2 + \beta \underline{w}) B(P\alpha) + \frac{k}{x_n} A(Q'\alpha) + \frac{k}{x_n} \underline{w} B(Q'\alpha)$$

consequently,

$$\frac{\partial}{\partial x_n}A(Q'\alpha) = -\beta A(P\alpha) - i\|\underline{w}\|^2 B(P\alpha) + \frac{k}{x_n}A(Q'\alpha), \qquad \frac{\partial}{\partial x_n}B(Q'\alpha) = iA(P\alpha) - \beta B(P\alpha) + \frac{k}{x_n}B(Q'\alpha). \tag{8}$$

Analogously, one has

$$\frac{\partial}{\partial x_n}P\alpha = \frac{\partial}{\partial x_n}A(P\alpha) + \underline{w}\frac{\partial}{\partial x_n}B(P\alpha) = -\beta A(Q'\alpha) - i\|\underline{w}\|^2 B(Q'\alpha) + i\underline{w}A(Q'\alpha) - \beta\underline{w}B(Q'\alpha)$$

obtaining

$$\frac{\partial}{\partial x_n}A(P\alpha) = -\beta A(Q'\alpha) - i\|\underline{w}\|^2 B(Q'\alpha), \qquad \frac{\partial}{\partial x_n}B(P\alpha) = iA(Q'\alpha) - \beta B(Q'\alpha). \tag{9}$$

In the case $\underline{w} \neq 0$ the general solution of the system (8) and (9) is

$$\alpha(\underline{w}, x_n) = x_n^{\frac{k+1}{2}} C_4^*(\underline{w}) \left(K_{\frac{k+1}{2}}\left((\beta + \|\underline{w}\|)x_n\right) + e_n K_{\frac{k-1}{2}}\left((\beta + \|\underline{w}\|)x_n\right) \right)$$

■

Notice that this representation also remains valid for $w = 0$ due to the proper asymptotic behavior of the Bessel K functions at zero.

In particular, one obtains that every solution of the equation (1) on the upper half-space that is $(n-1)$-fold periodic with respect to the orthonormal standard lattice $\Lambda^{n-1} = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{n-1}$ has a discrete Fourier series representation of the form

$$f(z) = \frac{1}{(2\pi)^n} \sum_{\underline{w} \in \Lambda^n} \alpha(2\pi\underline{w}, x_n) e^{-2\pi i \langle \underline{w}, \underline{x} \rangle}. \tag{10}$$

where $\alpha(2\pi\underline{w}, x_n) = ((\beta + 2\pi\|\underline{w}\|)x_n)^{\frac{k+1}{2}} \phi(2\pi\underline{w}) \Big(K_{\frac{k+1}{2}}\left((\beta + 2\pi\|\underline{w}\|)x_n\right) + e_n K_{\frac{k-1}{2}}\left((\beta + 2\pi\|\underline{w}\|)x_n\right) \Big)$.

This representation is valid for all functions that are solution to (1) on the upper half-space.

Growth orders and analogues of the maximum term and the central index

The aim is to develop analogues to some basic results from Wiman and Valiron's [15, 16] theory for solutions of (1) on the upper half-space. Under this point of view we introduce

Definition 1 *Let $f : H^+(\mathbb{R}^n) \to Cl_n$ be a solution of (1) with Fourier integral representation as described in Theorem 1. Then its order of growth and lower order of growth are defined respectively by*

$$\rho(f) := \limsup_{\|\underline{w}\| \to +\infty} -\frac{\|\underline{w}\| \log \|\underline{w}\|}{\log \|\phi(\underline{w})\|}, \qquad \lambda(f) := \liminf_{\|\underline{w}\| \to +\infty} -\frac{\|\underline{w}\| \log \|\underline{w}\|}{\log \|\phi(\underline{w})\|}. \tag{11}$$

If $\rho(f) = \lambda(f)$, then f is a function of regular growth. If $\rho(f) > \lambda(f)$, then f has irregular growth.

Remark: Due to this definition we may construct for arbitrary order of growth, examples of $(n-1)$-fold periodic functions which satisfies (1) using its discrete Fourier series representation on upper half-space.

We define for $(n-1)$-fold periodic solution of (1) with a discrete Fourier series of the form (10), the maximum term by

$$\mu(x_n, f) := \max_{\underline{w} \in \Lambda^{n-1}} \Big\{ \|\alpha(2\pi\underline{w}, x_n)\| \Big\}.$$

The notion of the central index $\nu(x_n, f)$ is equal to the value of $|\underline{w}|$ where $\underline{w}$ is (are) exactly that (those) discrete lattice point(s) for which the equality $\|\alpha(2\pi\underline{w}, x_n)\| = \mu(x_n, f)$ is attained.

In this context, we are much closer to the theory of entire monogenic functions, in which one considers discrete Taylor series. The lattice points $\underline{w}$ then correspond to the multi-indices $\mathbf{m} \in \mathbb{N}_0^{n-1}$ over which the summation of the multiple Taylor series is extended.

Proposition 1 *Suppose that $f : H^+(\mathbb{R}^n) \to Cl_n$ is a solution of (1) and $(n-1)$-fold periodic with respect to the orthonormal lattice Λ^{n-1}. Then there exists an $X_n > 0$ such that $\mu(x_n, f)$ decreases for $x_n \leq X_n$ strictly monotonic.*

In this result, analogous to the one given in [6], one can see that $\mu(x_n, f)$ is strictly monotonic decreasing. This is an important qualitative difference to the case of dealing with entire monogenic function whose maximum term $\mu(r, f)$ tends to infinity if r tens to infinity.

Using Proposition 1, one proves an upper bound estimate of the maximum modulus which we regard as a generalization of the classical Valiron inequality for holomorphic entire functions.

Theorem 2 *Suppose that f is an $(n-1)$-fold periodic solution of (1) on $H^+(\mathbb{R}^n)$ with period lattice Λ^{n-1}. Then there exists real constants L and X_n such that for all $0 < y_n < x_n \leq X_n$:*

$$M(x_n, f) \quad \leq \quad \mu(x_n, f)\Big((2\nu(x_n, f) - 1)^n + 2^n L \left(\frac{x_n}{y_n}\right)^{\frac{k-1}{2}} e^{-\frac{2\pi}{\sqrt{n}}\nu(y_n, f)(y_n - x_n)} \left(1 - e^{-\frac{2\pi}{\sqrt{n}}(x_n - y_n)}\right)^{-n}\Big).$$

Notice that this inequality does not depend on the particular choice of β.

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